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VALENCE FORMULA FOR THE CONGRUENCE GROUP $\Gamma_0(p)$

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Abstract. We prove the valence formula for the congruence group $\Gamma_0(p)$.

1. Introduction and statement of a main result

Let $\mathbb{H}$ be the complex upper half plane. Let $X_0(1)$ be the compact Riemann surface obtained by adjoining the the cusp ∞ to Riemann surface $SL_2(\mathbb{Z})\backslash\mathbb{H}$. For $\tau \in \mathbb{H} \cup \{\infty\}$, let $Q_\tau \in X_0(1)$ be the point associated to τ under usual identification. Then the following formula [3, Theorem 1.29] which is called a valence formula is well known: For a non-zero meromorphic modula form f of weight k on $SL_2(\mathbb{Z})$,

$$\sum_{Q_\tau \in SL_2(\mathbb{Z})\backslash\mathbb{H}} \frac{\mathrm{ord}_\tau f}{l_\tau} + \mathrm{ord}_\infty f = \frac{k}{12},$$

where $\mathrm{ord}_\tau f$ is the standard order of vanishing of f at τ, $\mathrm{ord}_\infty f$ is the order of vanishing of f at ∞ and $l_\tau \in \{1,2,3\}$ is the order of the isotropy subgroup of τ in $SL_2(\mathbb{Z})/\{\pm I\}$. This valence formula is useful in the theory of modular forms. Using the formula one can show that the dimension of the vector space of modular forms of weight k on $SL_2(\mathbb{Z})$ is less than or equal to $[k/12]+1$.

For a positive integer N, let

$$\Gamma_0(N) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) \;\middle|\; c \equiv 0 \bmod N \right\},$$

which is called the congruence group of level N. Note that $\Gamma_0(1) = SL_2(\mathbb{Z})$. In [2, Proposition 3.1], the author proved a valance formula for the congruence group $\Gamma_0(2)$. In this paper we generalize this formula to higher level cases. Let p be a prime and $X_0(p)$ be the compact Riemann

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surface obtained by adjoining the two cusps 0 and ∞ to Riemann surface $\Gamma_0(p)\backslash\mathbb{H}$. For $\tau \in \mathbb{H} \cup \{0, \infty\}$, let $Q_\tau \in X_0(p)$ be the point associated to τ under usual identification. With these notations we state the main result.

THEOREM 1.1. *Let p be a prime and f be a non-zero meromorphic modular form of weight k on $\Gamma_0(p)$. Then we have*

$$\sum_{Q_\tau \in \Gamma_0(p)\backslash\mathbb{H}} \frac{ord_\tau f}{l_\tau} + ord_0 f + ord_\infty f = \frac{(p+1)k}{12},$$

where $l_\tau \in \{1, 2, 3\}$ is the order of the isotropy subgroup of τ in $\Gamma_0(p)/\{\pm I\}$.

REMARK 1.2. A valance formula for $\Gamma_0(p)$ is well-known as in Theorem 1.1. But the proof of it does not appear in the literature. So we give Theorem 1.1.

2. The Proof of Theroem 1.1

For any function f on the complex upper half plane $\mathbb{H}$ and $k \in \mathbb{Z}$, the usual slash operator $|_k$ is defined as follows: For $\gamma := \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{R})$ and $z \in \mathbb{H}$,

$$(f|_k\gamma)(z) := (cz + d)^{-k} f(\gamma z).$$

Let N be a positive integer and $W_N := \begin{pmatrix} 0 & -\frac{1}{\sqrt{N}} \\ \sqrt{N} & 0 \end{pmatrix}$.

Let p be a prime and f be a non-zero meromorphic modular form of weight k on $\Gamma_0(p)$. Put $h := \mathrm{ord}_\infty f(z)$, $h' := \mathrm{ord}_0 f(z)$, and $q := e^{2\pi i z}$. Then f has the following expansions at the cusps:

$$\begin{aligned} f(z) &= \sum_{n \geq h} a(n) q^n \qquad \text{at } \infty, \\ (f|_k W_1)(z) = \left(f|_k \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \right)(z) &= \sum_{n \geq h'} b(n) q^{n/p} \qquad \text{at } 0. \end{aligned}$$

We define the theta-operator by

$$\theta f(z) := \frac{1}{2\pi i} \frac{d}{dz} f(z) = \sum_{n \geq h} n a(n) q^n$$

and denote by $E_2(z)$ the weight two Eisenstein series

$$E_2(z) := 1 - 24\sum_{n=1}^{\infty}\sum_{d|n} dq^n.$$

Note that we have

$$\frac{\theta f}{f}(z) = h + \sum_{n\geq 1} c(n)q^n \;\; (= h + O(q)).$$

By [1, Lemma 2.2], we have a meromorphic modular form G of weight two on $\Gamma_0(p)$ defined as follows:

$$G(z) := \frac{\theta f}{f}(z) + p\frac{\frac{k}{12} - h}{p-1}E_2(pz) + \frac{h - \frac{pk}{12}}{p-1}E_2(z) \;\; (= O(q)). \tag{2.1}$$

By [1, (2.17)], we obtain

$$\begin{aligned}(G|_2W_p)(z) &= \frac{\theta(f|_kW_p)}{f|_kW_p}(z) + \frac{\frac{k}{12} - h}{p-1}E_2(z) + \frac{h - \frac{pk}{12}}{p-1}p \cdot E_2(pz) \\ &= h' + h - \frac{(p+1)k}{12} + O(q).\end{aligned}$$

Since $(G|_2W_p)(z) = (\sqrt{p}z)^{-2}G\left(-\frac{1}{pz}\right)$, we have

$$(G|_2W_1)(z) = \frac{1}{p}(G|_2W_p)(z/p).$$

Hence the constant term of $(G|_2W_1)(z)$ is

$$\frac{1}{p}\left(h' + h - \frac{(p+1)k}{12}\right). \tag{2.2}$$

It follows from (2.1) and (2.2) that G has the following expansions at the cusps:

$$\begin{aligned}G(z) &= \sum_{n\geq 1} a_1(n)q^n && \text{at } \infty, \\ (G|_2W_1)(z) &= \frac{1}{p}\left(h' + h - \frac{(p+1)k}{12}\right) \\ &\quad + \sum_{n\geq 1} b_1(n)q^{n/p} && \text{at } 0.\end{aligned} \tag{2.3}$$

As before, we let $X_0(p)$ be the compact Riemann surface obtained by adjoining the two cusps 0 and ∞ to Riemann surface $\Gamma_0(p)\backslash\mathbb{H}$. We consider an abelian differental $w := G(z)dz$ on $X_0(p)$. For $\tau \in \mathbb{H} \cup \{0, \infty\}$, let $Q_\tau \in X_0(p)$ be the point associated to τ under usual identification. Then it follows from [1, (2.5), (2.6)] and (2.3) that we have

$$\begin{cases} \mathrm{Res}_{Q_\infty} w = 0, \\ \mathrm{Res}_{Q_0} w = \frac{1}{2\pi i}\left(h' + h - \frac{(p+1)k}{12}\right), \\ \mathrm{Res}_{Q_\tau} w = \frac{1}{l_\tau}\mathrm{Res}_{z=\tau} G \ \ (\tau \in \mathbb{H}), \end{cases}$$

where $l_\tau \in \{1, 2, 3\}$ is the order of the isotropy subgroup of τ in $\Gamma_0(p)/\{\pm I\}$.

Since $E_2(z)$ is holomorphic on $\mathbb{H}$ and $\frac{\theta f}{f}(z) = \frac{1}{2\pi i}\frac{f'}{f}(z)$, we have that for $\tau \in \mathbb{H}$,

$$2\pi i \mathrm{Res}_{z=\tau} G = 2\pi i \left(\mathrm{Res}_{z=\tau} \frac{\theta f}{f}(z)\right) = \mathrm{ord}_\tau f.$$

By a fundamental fact [1, p. 793] that $\sum_{Q\in X_0(p)} \mathrm{Res}_Q w = 0$, we obtain

$$0 = 2\pi i \sum_{Q\in X_0(p)} \mathrm{Res}_Q w = h' + h - \frac{(p+1)k}{12} + \sum_{Q_\tau \in \Gamma_0(p)\backslash\mathbb{H}} \frac{\mathrm{ord}_\tau f}{l_\tau},$$

which implies

$$\sum_{Q_\tau \in \Gamma_0(p)\backslash\mathbb{H}} \frac{\mathrm{ord}_\tau f}{l_\tau} + \mathrm{ord}_0 f + \mathrm{ord}_\infty f = \frac{(p+1)k}{12}.$$

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